Substituting technologies to reduce emissions over an infinite time horizon.

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1 Introduction.

Societies are increasingly concerned with emissions abatements problems and, although the proposition that current production of green house gases is unsustainable is now widely accepted, there is considerable debate about how reductions should be made. Although there is no doubt that we have to replace emissions producing technologies with technologies that provide for, possibly, increased levels of world consumption without the attendant risks of climate instability there are a number of alternative paths with strikingly different characteristics. For example the characteristics of wind and solar as possible replacements for fossil fuel sources of energy are not the same as those of nuclear. This raises a number of questions. These have considerable policy relevance as planners try to consider long term replacement schedules and are subject to ongoing debate. Much of this debate is economic and political and goes to issues of risks and resources and preferences. On the other hand there are also some basic questions concerning the optimal trajectory for any replacement schedule and the effect of different technologies on this.

The purpose of this paper is to look at some of these issues. It is specifically concerned with questions about the optimal trajectory for replacing the old technology over time. For example, if a more powerful technology came available is it best hold the trajectory of emissions reductions constant and increase consumption or to split the difference and allow emissions reductions and consumption to both increase? Should consumption be held constant with an even faster reduction in emissions?

The answer to these questions only covers a few aspects of the issues in contention. Nonetheless it is hard to get a clear intuition on them and additional clarity does give us some clues as to the best course of action under certain conditions.

In order to dealt with these questions optimal trajectories are examined over an arbitrarily long time period. This raises some analytical issues concerning uncertainties about the damage of destabilizing the climate and dealing with future returns.

From what we know of the climate science the range of potential damage in the stock of emissions is large and increasing in a non linear manner. Moreover the risks are all downside. In order to make sense of the problem it will be assumed that the rate of increase in damage resulting from emissions stock is large. It will also be assumed that the discount rate is sufficiently small to make the future count. I will specify these ideas more precisely during the analysis.

In summary the analysis shows that the optimal trajectory for the replacement programme is to begin by reducing consumption and to continue on this path until most of the old technology is replaced and to then accelerate consumption to reach a stable state, if this exists, where it is then maximized for all time. It also shows that, as the capacity of the replacement technology to displace the old improves, more consumption is sacrificed in the early stages to speed up the rate of technology replacement. This is somewhat unexpected. It might have been guessed that the improved technology would be used to increase both consumption and the replacement rate.

If costs increase significantly with the capacity of the new technology things are more complex. From a policy perspective, however, there is still an argument, in most cases, for pushing the technology with the greatest capacity and making the appropriated reductions in consumption to accelerate emissions reduction.

I set out the paper in three sections. In the second section I state the problem more precisely and generate a simple dynamic model. In the third the trajectories are studied. In the forth I look at the question of different technologies.

2 The problem and model

2.1. The problem

Suppose there is a planner who wants to maximize an index of welfare out to some indefinite time in a system with a fixed stock of technology that produces an all purpose good that can be used either for consumption or to produce a new technology. The existing technology also produces a flow of emissions. The stock of these emissions inflict an increasing cost in terms of environmental destruction. The new technology can replace the old technology and can produce the same good with zero emissions. There are different possibilities for this technology and these may replace the old technology at different rates. For example one unit of technology A may displace one unit of the old technology whereas it might require two units of technology B to do the same job. It is possible that the new technology could also be used to reduce existing stocks of emissions by scrubbing the atmosphere. Although this is an interesting possibility I ignore it for present purposes.

2.2. The model

The model is specified by letting $c(t)$ be the level of consumption and $x(t)$ be the stock of emissions at time t. Utilities from consumption are $w(c)$ and the losses from emissions are $f(x)$. The discount rate is given by the positive constant $\delta < 1$. In keeping with the discussion in the introduction this is arbitrarily small. It is assumed that w and f are continuously differentiable. Utilities from consumption have the usual form

$w_c > 0$, $w_{cc} < 0$. It is assumed that $f_x > 0$ and $f_{xx} > 0$. It would be consistent with what is now thought to be the case with climate change that f could have the characteristics of a penalty function. I will explore different possibilities.

The function that the planner needs to optimize can be written as

$$J = \int_0^\infty e^{-\delta t}(w(c) - f(x))dt \tag{1}$$

subject to the constraints on the dynamics of the system.

The stock of the existing technology is k and this produces a constant amount k of the all purpose good in each time interval and also a flow of emissions measured at k units per time interval. The stock of alternative technology at time t is $m(t)$. This replaces β units of the old technology where $\beta > 0$ and produces β units of the all purpose good. If for example it takes two units of new technology to replace one unit of the old then $\beta = \frac{1}{2}$. Since emissions from the new technology are zero

$$\dot{x} = k - \beta m \tag{2}$$

and it will be assumed that $\beta = 1$ for the time being. I will vary this later. Since I am not going to investigate the possibility that emissions are scrubbed from the atmosphere we need the condition

$$k - m \geq 0$$

for all t.

The stock of new technology is produced from the all purpose good and the cost of this may depend on the technology chosen. The cost parameter is $\sigma(\beta)$. For example if we take a reference cost which produces one unit of the new technology for one unit of the all purpose good then $\sigma = 0$. This gives

$$\dot{m} = k - (1 - \sigma(\beta))c \tag{3}$$

with $m(0) = 0$ and since $\beta = 1$ to begin with is our reference technology we set $\sigma(1) = 0$.

The initial level of consumption is fixed and the dynamics are given by

$$\dot{c} = \alpha \tag{4}$$

where $c(0) > 0$ and the control $\alpha(t)$ is bounded by $[\alpha \in [-h, +h]$. Since consumption cannot exceed output

$$k - c \geq 0$$

for all t.

It is also necessary to exclude a solution that makes consumption negative in order to accelerate technology replacement

$$c \geq 0$$

3 Optimal trajectories

3.1 Strategy for a solution

The strategy is to assume a turnpike solution that gives a stable state after some finite time period exists and to derive this by using the Pontryagin maximum principle. The Hamiltonian is

$$H = e^{-\delta t}(w(c) - f(x) + p(k - m) + q(k - c) + v\alpha)$$

where we are continuing to assume $\beta = 1$ and $\sigma = 0$. We now have the Lagrangian

$$L = H + \theta_1(k - m) + \theta_2(k - c) + \theta_3 c$$

where $p(t), q(t), v(t)$ are costate functions and are almost everywhere continuously differentiable. The necessary conditions that an optimal trajectory must satisfy are that α maximizes H subject to

$$\begin{aligned} H_\alpha &= v \\ \dot{p} &= \delta p + f_x \\ \dot{q} &= \delta q + p + \theta_1 \\ \dot{v} &= \delta v - w_c + q + \theta_2 - \theta_3 \end{aligned} \tag{5}$$

and any trajectory that satisfies these will be a solution if it also satisfies the Mangasarian sufficiency conditions.[1] These are that there exists a continuous p, q and v such that

$$\begin{aligned} lim_{t\to\infty} e^{-\delta t} px &\geq 0 \\ lim_{t\to\infty} e^{-\delta t} qm &\geq 0 \\ lim_{t\to\infty} e^{-\delta t} vc &\geq 0 \end{aligned} \tag{6}$$

where x, m and c are admissible trajectories and H is concave or linear in α, x, m and c.

3.2. A consumption maximizing trajectory

The problem of maximizing H with respect to α is the same as maximizing with respect to c with the addition of the condition that $c(0)$ is fixed. Call the problem of optimizing when $c(0)$ can be freely chosen the $\bar{c}$ problem. The solution to the α problem is in a subset of permissible trajectories for the $\bar{c}$ problem. It follows that a solution to the α problem must satisfy the conditions for the trajectory that gives a solution to the $\bar{c}$ problem when these conditions can be attained. A solution to the $\bar{c}$ problem must satisfy

$$w_{\bar{c}} = q + \theta_2 - \theta_3 \tag{7}$$

and let this be given for $\bar{c}(0) = \hat{c}$

[1] See ([15], 234-6)

Consider the trajectories that might satisfy the required conditions for the $\bar{c}$ problem:
$[a]$ $c = 0$ until $t = \bar{a}$ and then jumps to satisfy $c = k$. To show this is not an optimum let $\bar{a} = supt : c = 0$. Consider a path with a small perturbation that sets $c > 0$ in some small interval $(\bar{a} - \epsilon, \bar{a})$. As $\epsilon \to 0$ the change in the Hamiltonian is

$$H_c = \epsilon(w_c + (-f_x \frac{dx}{dm} - p + p_x \frac{dx}{dm}(k - m) - q + q_x \frac{dx}{dm}(k - c))\frac{dm}{dc})$$

and since $\frac{dw}{dc} \to \infty$ when evaluated at $c = 0$ and q, p and $\frac{dx}{dm}\frac{dm}{dc}$ are finite we have $H_c > 0$ and hence $c = 0$ is not an optimal trajectory. It follows by a similar argument that $c \neq 0$ in $(\bar{a}-(n+1)\epsilon, \bar{a}-n\epsilon)$ for $(n+1)\epsilon \leq \bar{a}$.

This leaves

[b] $c > 0$ for all t.

Trajectory $c > 0$. The trajectory terminates at $\bar{a}$ and from equation (5) we have $\dot{p} \to 0$ as $t \to a$ and hence

$$p = -\frac{f_x}{\delta} < 0$$

and to approach this continuously $p < -\frac{f_x}{\delta}$ and $\dot{p} < 0$ in the vicinity of $t = \bar{a}$. If $p > -\frac{f_x}{\delta}$ then $\dot{p} > 0$ and the terminal condition cannot be satisfied. This means

$$p < 0 \quad \text{and} \quad \dot{p} < 0$$

for all $t < a$.

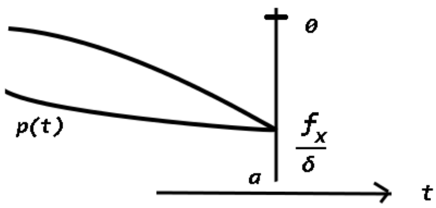

Figure 1. Example of the trajectory for the costate p

It is not necessary that $\dot{q} \to 0$ as $t \to \bar{a}$ since $\dot{q}(a) = \delta q + p + \theta_1 = 0$ from equation (5). It will be observed that since $\theta_1 \geq 0$ the value for $\dot{q}$ can only jump up at $t = \bar{a}$ when θ_1 becomes active. This means that

$$\dot{q} \leq 0$$

at $t = \bar{a}$.

For $\dot{q} \leq 0$ at $t = \bar{a}$ there are two possibilities.

(i) $q = -\frac{p}{\delta}$ and hence $\dot{q} = 0$ with $\theta_1 = 0$ at $t = \bar{a}$. Since $q < -\frac{p}{\delta}$ for $t < \bar{a}$ means $\dot{q} < 0$ the end point is unobtainable and hence $q > -\frac{p}{\delta}$ and $\dot{q} > 0$ for $t < a$. Since $\dot{c} = \frac{\dot{q}}{w_{cc}}$ it follows that $\dot{c} < 0$. It will be noted from equation (7) that at the terminal time $\bar{a}$ the Lagrangian multipliers do not provide a mechanism for $c(\bar{a})$ to jump up. Since $c < k$ for all t it follows that $c(\bar{a}) < k$. This means that the $\dot{m} = 0$ at $t = \bar{a}$ cannot be satisfied.

(ii). $q < -\frac{p}{\delta}$ at $t = a$. There are two ways this might be satisfied.

(iia). $q < \frac{p}{\delta}$ for all $t < a$. In this case $q > 0$ and $\dot{q} < 0$ all $t < \bar{a}$.

(iib). $q > \frac{p}{\delta}$ for $t < b < a$ and $q < \frac{p}{\delta}$ for $b > t < \bar{a}$. In this case there is a turning point at $t = b$ where $\delta q = -p$. We have $\dot{q} > 0$ for $t < b$ and $\dot{q} < 0$ for $b < t < \bar{a}$.

See fig. 2. Both trajectories are feasible for the $\bar{c}$ problem under different conditions.

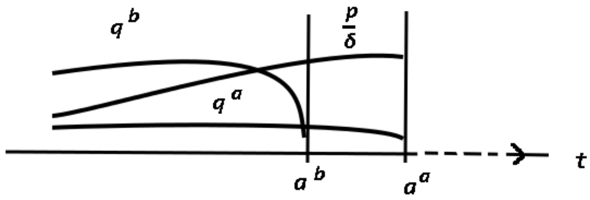

Figure 2. Example of possible trajectories for costate q

In most problems of this type the optimal trajectory is to move to the turnpike as quickly as possible. In this case (iib) should be chosen. I will consider the conditions under which this is the case.

In trajectory (iia) we have $\dot{c} = \frac{\dot{q}}{w_{cc}} > 0$. In trajectory (iib) we have $\dot{c} > 0$ for $t < b$ and $\dot{c} < 0$ for $b < t > \bar{a}$. Let the time at which the system reaches a stationary point for (iia) be $t = a^a$ and for ii^b be $t = a^b$. For some time $t \in (0, A)$ where $A < a^a$ we have $q^b > q^a$ and hence $c^b < c^a$ and $m^b > m^a$. It follows that $m^b = k$ at time $t = a^b$ and $m^a = k$ at $t = a^a > a^b$. This means that $f^b < f^a$ for all $t > 0$. It now follows for δ sufficiently small

$$\int_0^t (w^b - f^b) ds > \int_0^t (w^a - f^a) ds$$

for t sufficiently large and (iib) is the optimal programme.

It is worthwhile noting that the discount rate provides the simplest analytical tool for this argument, but other conditions would have given the same result. Among these are that f_{xx} or w_c at $c = k$ is sufficiently large. In the first case we can choose an f such that the the cumulative difference $f^b - f^a > 0$ for any δ. In the second case we can choose a $w : q = w_c(k)$ is not attainable in programme (iib).

It also follows that, if sufficiently restrictive conditions are placed on δ, f and w, trajectory (iib) will be optimal using the overtaking criteria.

In summary a programme that decreases consumption and accelerates it near the stationary state is optimal under a variety of conditions. A programme that increases consumption for the entire period to the stationary state is optimal only if: (a) the future is heavily discounted, or (b) if the rate of increase in damage from emissions is small and the marginal utility from from consumption in the stationary state is small.

3.3. Trajectory for $c(0)$ fixed

Where the starting point cannot be freely chosen the optimum trajectory will be to move to the path that maximizes c as soon as possible. Assume that $c(0) > \bar{c}(0)$. Then

$$w_c < w_{\bar{c}} \quad \text{and} \quad -w_c + q > 0$$

in equation (5). In order to move to the optimum path let $v(0) < 0$ such that

$$v(0) - w_c + q > 0$$

and hence

$$\alpha = -h \quad \text{and} \quad \dot{v} > 0$$

from equation (5). It follows that c decreases at the maximum rate permissible until $-w_c + q = 0$ at some $t = T$.

To stay on this trajectory we need $v = 0$ for all $t > T$ and hence α can be freely chosen to set $c : -w_c + q = 0$ for all $t \in (T, a)$. Using this value for v and α we have $\dot{v} = 0$ for all $T < t < a$, as required. See fig. 3.

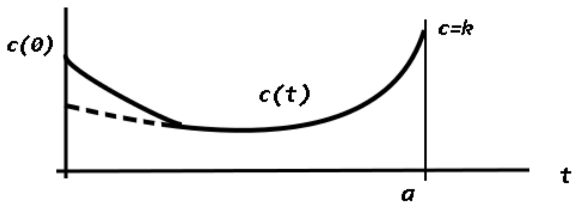

Figure 3. Trajectory with the programme for a low discount rate

Since the initial level of consumption is greater than the level that would be freely chosen the terminal time a is not necessarily the same as $\bar{a}$ and the terminal stock off emissions may be greater than in the $\bar{c}$ programme.

4 Trajectory with changes in technology and cost

4.1. Changing technologies

Consider the case where there are different technologies that may have varying degrees of effectiveness in displacing the old technology. In this case β in equation (2) is allowed to vary. Changes in cost are ignored for the time being and $\sigma = 0$. This will be relaxed in the next section. If we solve the differential equation in (5) we get

$$q(t) = e^{\delta t}(-\beta \int_t^a e^{-\delta s} p ds + q(a^\beta)e^{-\delta a}) \tag{8}$$

where a^β indicated the possibly different terminal time.

Differentiating with respect to β gives

$$q_\beta = e^{\delta t}(-\int_t^a e^{-\delta s} p ds) \tag{9}$$

and hence

$$q_\beta \geq 0$$

with $q_\beta(a^\beta) = 0$ as required to satisfy the condition $q : c = k$ and hence $q(a^\beta) = q(a)$.

We also have

$$\dot{q}_\beta < 0$$

for all $t \in (b^\beta - \epsilon, a^\beta)$.

It follows that

$$c_\beta < 0 \quad \text{and} \quad \dot{c}_\beta > 0$$

In other words the planner responds to an increase in the efficacy of the technology by decreasing consumption at the outset of the programme. Since the rate of decrease goes to zero as $t \to a^\beta$ the reduction is greatest at the beginning of the programme for $t < b^\beta$.

It will be noted that since $c^\beta < c$ we have $a^\beta < a$. In order to achieve this the new programme must cross the trajectory of the old programme at some point $t < a^\beta$.

4.2. Changing costs

Consider the case where cost depends on the technology. It might be imagined, for example, that weaker technologies are less expensive although it turns out that the evidence for this is not clear. Part of the issue is that weaker technologies incur a range of related costs such as the need for back-up facilities and modifications to the grid. For purposes of the analysis I will simply assume that the stronger technology is more expensive. If it less expensive then the results are obvious.

If we set $\sigma > 0$ in the Hamiltonian the trajectory for the $\bar{c}$ problem becomes

$$w_c = (1-\sigma)q$$

and we can differentiate with respect to β to get

$$w_{c\beta} = q_\beta(1-\sigma) - q\sigma_\beta$$

From equations (8) and (**??**)

$$q_\beta = \frac{1}{\beta}(q - e^{\delta(t-a)}q(a))$$

which gives

$$w_{c\beta} = \frac{1}{\beta}q(1-\sigma-\beta\sigma_\beta) - \frac{1}{\beta}(e^{\delta(t-a)}q(a))$$

For a sufficiently large we can ignore the last term for t small. Hence

$$w_{c\beta} > 0 \quad \text{if} \quad (1-\sigma-\beta\sigma_\beta) > 0 \tag{10}$$

for t close to zero.

It follows that there are two possibilities:
(i) beginning at $\sigma = 0$ and $\beta = 1$ if $\sigma_\beta < 1$ we have $w_{c\beta} > 0$ and c is reduced as β is increased for t close to zero. As $t \to a$ we have $w_{c\beta} < 0$ with the terminal condition $c = \frac{k}{(1-\sigma)} > k$. In other words the trajectory of c with the new value for β and σ starts below the old trajectory and crosses it as $t \to a$ to give a higher terminal value. See fig. 4.
(ii) if $\sigma \to 1$ or $\sigma_\beta > 1$ we have $w_c < 0$ and the trajectory of c starts above the old trajectory and continues above it to give higher terminal value.

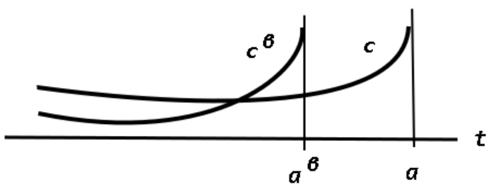

Figure 4. Examples of consumption paths for an increase in β

4.3. An optimal choice of technology and cost

A natural question at this stage concerns the optimal choice of technology. Although the question is obvious the solution is a little difficult. In order to think about it treat the optimal trajectory to the stationary state at $t = a$ as a problem with an endpoint that depends on β and a scrap value to represent the value of the payoff in the period $t \in (a, \infty)$. Let the scrap value be $\varphi(w, f)$ where w and f are implicit functions of a. In this problem a is optimally chosen

If an optimal choice of technology exists it must satisfy the necessary condition ([10], 255-6)

$$\int_0^a (-pm + qc\sigma_\beta)dt = a_\beta \varphi_a$$

using the fact that the condition for a free terminal time scrap value problem is $H(a) + \varphi_a = 0$

It will be noted that $\varphi_a < 0$ since a longer time to the stationary state allows emissions to increase. It follows that, if there is an optimal choice of technology it must satisfy the condition

$$a_\beta < 0$$

because the integral term is positive.

It follows that if there is an optimal choice of technology it must be in a region that decreases consumption for some time along the optimal trajectory.

4.4. Note on existence

A practical question is does an optimal choice of technology always exist? To answer this note that it has so far been assumed that the reference technology has $\beta = 1$ and $\sigma(1) = 0$. This is purely for convenience. If we relax these assumptions to allow the full range of technologies the answer is yes.

Let $\beta > 0$ and hence for $\beta < 1$ we have $\sigma < 0$ with $\sigma_\beta > 0$ as before. It follows that there is some value of β for a given σ such that $(1 - \sigma - \beta\sigma_\beta) > 0$ in equation (10). All we need to complete the argument is that $q(0) > q(a)$.

5 Remarks

The main message is that, if a stationary state exists and the rate of discount is sufficiently small, it is optimal to move as quickly as possible to a path which continually reduces consumption until most of the old technology has been replaced. After this time consumption is increased to its maximum and stays at this zero cost in emissions for all future time.

As the ability of the new technology to displace the old, increases consumption is reduced along the entire trajectory to some time close to the new stationary point $t = a^\beta$. This decrease occurs most rapidly at the start of the programme. At some point close to a^β consumption is then accelerated to reach its maximum value. This result holds for the low discount rate trajectory of most interest and also for the alternative trajectory.

Why aren't the gains from technology shared or split between consumption and the stock of emissions? One way to think about this slightly counter intuitive result is to consider the adjoint variable q as the shadow price of consumption at every point. Observe that what the stronger technology does is to increase the value of q. In other words it increases the opportunity cost of consumption relative to emissions. From this perspective the result then follows from orthodox theory.

If costs are taken into account the analysis becomes more difficult but there are some interesting insights. If, as is plausibly the case costs are roughly equivalent across technologies then the optimum policy is always to adopt the stronger replacement technology.

If it is assumed that costs increase at a sufficiently high rate with the strength of the replacement technology we see that there is an internal solution to the best choice of technology. This has the characteristic that the choice should be made in a region where the total effect of the cost increase and the stronger technology is to reduce the time required to reach the stationary point.

In practical terms this tells us that, if we are concerned with long term payoffs then there is a argument for pushing the technology with the greatest displacement potential unless there are significant cost penalties. In addition this should be supported by the slightly counter-intuitive measure of reducing consumption to speed up emissions reduction.

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