The political-economy of production and distribution: a non-atomic game

Abstract

The aim of this paper is to investigate some questions about the macro-logic of the distribution of produced goods when agents have political and economic power. It attempts to reveal some aspects of these issues not covered by the more usual rational choice approaches. It does this by constructing a model with an arbitrarily large number of agents treated as a continuum and allowing distribution to be determined by production and by competition between any possible political coalitions. It explores such questions as: How would the pattern of distribution change as the proportion determined by political power changes? How is this pattern affected by the initial distribution of endowments? What sort of agents benefit most and when?

Alex Coram. Economics. U. Mass. Draft. Sept o13

1 Introduction

Societies are often compared in terms of differences in the shares of distribution controlled by political mechanisms and the shares controlled by market forces and there is extensive discussion of the effects of these differences on things like total output and economic participation. So far much of the analysis of these questions has been carried out at an historical and empirical level. Despite the detail and depth of the studies they still leave a number of questions that are best thought of in theoretical terms. Among these are: what would be the properties of an allocation if every agent had the same political power but different productive capacities? How would the allocation change when the share of distribution controlled by the political system changes? Who benefits and who loses? What happens when people can choose whether or not to work? How does the final distribution of goods differ as the initial distribution of endowments becomes more or less egalitarian? How do changes in the initial distribution of endowments affect the amount produced and the sensitivity of the system to changes in the share allocated by political decision? What happens to total welfare as the proportion distributed by voting increases?

The problem with trying to answer these questions is that there are considerable difficulties in finding ways to integrate the political and the economic into a single analysis. This paper is an attempt to contribute to this task and to help provide some new perspectives on these issues.

The model is constructed by taking a top down approach to the political-economic system and it does not model political entrepreneurs or voters' at the level of individual optimization decisions over response and strategic competition. In this sense it moves away from an analysis in terms of micro-foundations to one that deals with a system made up of an arbitrarily large number of agents where a lot of individual differences wash out, as it were. This means we can think of the distribution to individuals as reflecting their potential worth to political coalitions and contribution to production.

In contrast, most of the literature concentrates on analyzing an equilibrium in a situation where a number of candidates or parties compete by offering an allocation of resources and voters support whichever allocation maximizes their utilities. This is usually done by implicitly or explicitly restricting the number of coalitions that can form either through limiting the number of parties or by imposing conditions on the tax schedules that can be offered.[1] Although these restrictions strengthen the analysis in some directions and allow us to use some powerful tools from optimization theory, they weaken it in others by leaving the implications of this large and shifting set of possible coalitions out of the picture. For example with 30 voters and a majoritarian voting rule there are about 150 million possibilities if we are prepared to accept that distributions can be sufficiently fragmented to appeal to each voter.

I do not offer any further justification for the approach taken in this paper except to note that there is a large literature which questions whether voters behave with the calculative rationality required for a complete micro-foundations approach.[2] For or purposes it is simply taken as an alternative to micro-foundations. It is not necessarily the case that one approach is better than another, of course. They just highlight different aspects of the distribution problem.

It is implicitly assumed that coalitions are put together by political entrepreneurs who attach the same value to each vote. This is consistent with aspects of the Downsian approach to political competition.[3] It differs in that distribution can be targeted specifically to small groups of voters and does not have to be allocated on a left right continuum.

It will be argued that the best tool for studying this distribution is the analogue of the Shapley value for continuous games. This is explained in more detail below.

Many of the theoretical foundations for this paper have been developed by Aumann and Shapley [1], Aumann and Kurz [3], Shapiro and Shapley [17] and Milnor and Shapely [10] and I rely heavily on their work.

In rough terms the answers to the previous questions are as follows.

i. The level of inequality between individuals decreases as the portion of the product allocated by the political system increases.

ii. The rate at which individuals choose not to work for any proportions of the total product distributed by the political system will depend on the initial level of inequality in initial endowments. It will also depend on the utilities associated with work at different endowment levels.

[1] See Roemer [14] for an example of a two party model with and good recent discussion of the literature on vote competition. Osborne and Slivinski [12] give a model where the number of candidates is not restricted but voter preferences are linear.

[2] See for example Shapiro and Green [18].

[3] See Downs [5] and for an example of a modern development Lindbeck and Weibull [9].

iii. If the spread in initial endowments is sufficiently unequal some individuals will choose not to work at arbitrarily low proportions of distribution by the political mechanism.
iv. As equality in initial endowments increases, the proportion of output distributed by by voting that is required for individuals to leave the workforce increases. On the other hand the rate at which they leave is higher than in less egalitarian systems.
v. For a sufficiently large spread in initial endowments an increase in political power initially increases total welfare.

Although the result that an increase in the proportion of the product distributed by political processes will lead to more equal outcomes might seem obvious after the event, some of the other findings about the macro-logic of the system are less intuitive. For example one implication of the results is that, as equality in initial endowments increases the levels of distribution through voting that can be sustained without loss in production also increases. Beyond some level, however, the more equal the initial distribution of endowments the more rapidly production is lost as the share of the product distributed by voting increases.

Such results would seem to suggest that more egalitarian systems are insensitive to an increase in a shift of distribution from market outcomes to government up to a point and then become extremely sensitive. Less egalitarian systems are likely to be sensitive to this shift though their whole range.

To the extent that these results and their corollaries could be translated into observable outcomes it should be possible subject them to testing, or at least to incorporate them into explanations of differences between systems such as the US and some European societies, for example. This is beyond the scope of this paper.

I set out the paper as follows. In §.2 the problem and the model are presented. §.3 gives the main theorem. This is analyzed in §.4.

2 The problem and model

2.1. Statement of the problem

Suppose we have a society with an economy described in terms of a production function, endowments of capital and labour capacity, and a political system described by a collective choice mechanism. The economy might be thought of as either a capitalist system in which each individual has an initial endowment of resources and skills or a system in which all resources are collectively owned and workers have different skills. Although the model is written in terms of capitalism it holds for both cases. This second some independent interest in the debate about distribution in the Lockean state of nature but is not pursued here. It is assumed that no individual has enough capital to extract rent and everyone has to work with their own endowments. The system might be thought of as some sort of lemonade stand capitalism. There is full employment in the sense that everyone who wants to work has a job. The collective choice rule says that each individual has equal political weight and any coalition containing some fixed proportion of votes greater than half is winning. A winning coalition can distribute some portion of the produced goods to its members. This share is fixed in the short run. The level at which it is fixed might be thought to characterize a particular system with a higher share representing higher levels of government spending on collective goods and greater welfare payments and the like. Individuals may or may not be forced to work in different cases to be examined.

The model is constructed by using a Lebesque measure on the interval I, with $\mu(I) = 1$, and treating each individual as an infinitesimally small sub-interval of I written dt or $\mu(i)$ or μ depending on the point to be made, where $\mu(i) = \mu(j)$ for all i, j. Coalitions are written S can be thought of either as sets, or as measurable subsets of I, depending on the circumstances. The size of a coalition is given by $\mu(S) = \mu(S \cap I)$. A coalition is winning when $\mu(S) \geq b \geq \frac{1}{2}$.

There is a single all purpose produced good and utilities in this good are linear. It is assumed that work may have some disutility. This is expressed in the same units as utilities in the produced good.

Individuals have different productive capacities corresponding to their endowments. These capacities can be thought of either as a vector or as a simple function g that takes on a finite number of values $(g^1, \ldots, g^q)$ on the interval I. Write the set of individuals with capacity g^r as G^r. We can think of $g^r dt$ as the the capacity of an individual in G^r. Since g is Lebesque integrable we can define a vector

$$\gamma(S) := (\int_{G^1 \cap S} g^1 dt, \int_{G^2 \cap S} g^2 dt, \ldots, \int_{G^q \cap S} g^q dt)$$

and let $\gamma^r(S) := \int_{G^r \cap S} g^r dt$. It will be useful to express productive capacities as a measure by defining

$$w^r(S) := \frac{\gamma^r(S)}{\sum_j \gamma^j(I)}$$

where $j = (1, \ldots, q)$. It follows that $w^r(S)$ is the proportion of agents with productive capacities r that are in S. This means

$$\sum_r w^r(I) = \mu(I) = 1 \tag{1}$$

It is now possible to think of the measures of productive capacities in two ways. The first is as a partition of the interval I with $w^r(i)$ belonging to the agent i in interval $\mu(i)$. See fig. 1b. The second is as a q dimensional vector space with a basis given by orthogonal vectors $w = (w^1, w^2, \ldots, w^q)$ as in fig. 1c. To avoid proliferating notation w^r will be used as either a vector or the measure of capacity $w^r(S)$.

The production function is defined over $\mathbf{R}^q$ and is assumed to be real valued, continuously differentiable and monotonically increasing. When everyone works it is

$$f(w)$$

with $f(0) = 0$. It is assumed that f can be expressed as a polynomial.[4]

[4]There isn't any loss here since the polynomials are dense in the space of continuously differentiable functions. See [4]

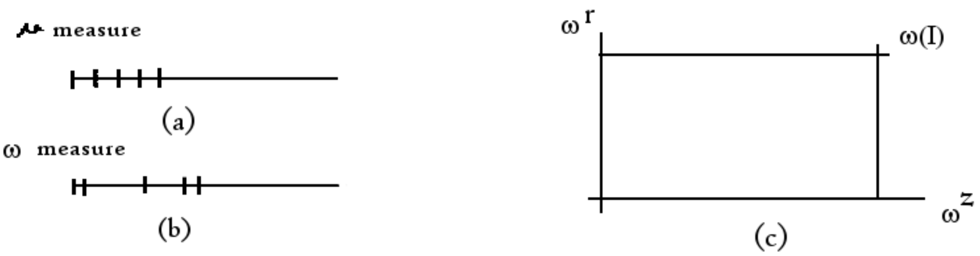

Figure 1. Measures on I and in $\mathbf{R}^q$

Let Ω be the set of all possible coalitions and S a subset of Ω. To deal with the payoff to a coalition define a monotonic increasing set function that gives the maximum that S can achieve under the decision rules and the production technology by

$$v : S \to \mathbf{R}$$

where $v(\phi) = 0$. Since the function v carries all the information about the value of any coalition it is referred to as the characteristic function or the game.[5] From the assumption of linearity v is given in utilities that can be transferred between its members.

Write the share of the product that can be distributed by the winning coalition as $a \in [0, 1]$. This parameter gives the portion of the total product allocated by governments and markets respectively and it would be possible to treat this as a crude index of political and economic power. In this way we can get a rough characterization of advanced political-economic systems along the lines discussed above.

Agents will only work if the returns from working are greater than the returns from not working. Let the effect of agents in coalition S not working on the productive capacity of S be given by $\psi(a)w(S)$. If everybody works $\psi = 1$. If no one works $\psi = 0$.

Each coalition will get the portion of the product that it captures as the result of its political power and the portion it obtains as the result of its productive capacity. Let ϵ_i or $\bar{\epsilon}_i$ be parameters that give the loss of utility from working depending on the context and assume $\epsilon_i = \epsilon_j = \epsilon$ for all i, j. Where everyone must work $\epsilon = \bar{\epsilon} = 1$. When individuals are not required to work $\{\epsilon, \bar{\epsilon}\} = \{1, 0\}$ for an agent that does not work and $\{\epsilon, \bar{\epsilon}\} = \{x, x\}$ for an agent that works for $x < 1$. This gives

$$v(S) = \left| \begin{array}{c} \epsilon^s(S)af \circ (\psi w(I)) + \bar{\epsilon}^s(S)(1-a)f \circ (\psi w(S)) \quad \text{when} \quad \mu(S) > b \\ \epsilon(S)(1-a)f \circ (\psi w(S)) \quad \text{otherwise} \end{array} \right. \tag{2}$$

where ϵ^s is a function that represents the utility for individuals in S working or not working.

[5]This is not to be confused with the usual characteristic function in mathematics.

2.2 The value for the game

The share of v that gives the value for the game to each agent is given by a function

$$\varphi_i : v \to \mathbf{R}$$

and it is possible to specify the way in which this share might be allocated in several ways that reflect the potential of an agent and its capacities. One allocation that is well established in the literature and has been extensively discussed is the Shapley value. For finite games this is given by

$$\bar{\varphi}_i[v] := E[\sum_{S \ni i}(v(S) - v(S - i))] \tag{3}$$

where E is the expectations operator when all orders are assigned equal probability.

If the value of the finite game converges to the value of the game with a continuum of agents, φ, it will be possible to use the properties of the continuous agent game directly. Convergence is difficult to prove but fortunately we already have the foundations.[6] It is only necessary to show that the required conditions can be satisfied.

2.3 Convergence

Theorem 1 *The value of the finite game v converges to the unique value of the continuous game if v can be expressed in linear powers of non-atomic measures.*

Proof: See Aumann and Shapley ([1], p.23) and [3] proposition 12.8.
□

We need to prove the following lemma.

Lemma 1: *$f \circ w$ can be written in linear powers of measures.*

Proof of lemma 1 : Since w is a measure in $\mathbf{R}^+$ for any fixed n and the product and sum of measures is a measure we only have to show that $f \circ w$ can be expressed in linear powers of products and sums of measures. It follows from the fact that f is a polynomial that the only terms which are not in powers of measures are of the form $w_1^i w_2^j w_3^k \ldots w_m^z$. It will be seen that $w_1^i w_2^j$ can be expressed in the form $\frac{1}{2}(w_1^i + w_2^j)^2 - \frac{1}{2}(w_1^{2i} + w_2^{2j})$. It follows by induction that $w_1^i w_2^j w_3^k \ldots w_m^z$ can be expressed as powers of partial sums and powers.
□

2.4. Rewriting the production function

Before commencing the analysis it is possible to make life easier by simplifying the production function, This is done using the idea that, if a coalition is sufficiently large, the proportion of different sorts of agents in its composition would be expected to approximate the proportion in the population as a whole. For a

[6]See Aumann and Shapley [1], Aumann and Kurtz [2, 3] and Shapiro and Shapley [17].

completely rigorous mathematical proof of this see [1].

Proposition 1. For a coalition of measure $\mu(S) = t$ there is an n sufficiently large that the probability $\{|w(S) - t(w(I))| \geq c\} \leq \delta$ for any $\delta > 0$.

Proof. This is a straightforward application of Tchebychev's inequality and since we are dealing with a continuum of agents we are done.
□

It follows from Proposition 1 that we can rewrite the production function for a coalition of measure $\mu(S) = t$ as

$$f \circ \psi\omega(S) = f \circ t\psi\omega(I) + o \tag{4}$$

where o is an error term that is vanishingly small.

3 The value theorem

3.1. The main theorem

The main theorem says that every agent gets a share determined by its political power and its marginal contribution to production. It follows the approach and some results in [1]. Because marginal returns are familiar in other contexts we forget that they are something of a surprise since there is nothing in the axioms that the Shapley value satisfies to indicate that this should be the case. It also gives the unexpected result that changes in the number of voters required to be decisive does not alter the value of the game to any individual.

3.2. Main theorem

Theorem 2 . *The value for the game v for an individual $i \in \mu(i)$ with productive capacity w^r is given by*

$$\varphi_i = \epsilon_i a f \circ (\psi w(I))\mu + \bar{\epsilon}_i(1-a)\int_0^1 f_{w^r} \circ (\psi t w(I))dt$$

where f_w is the directional derivative of f with respect to w.

Proof of Theorem 2. The proof is constructed by treating μ and w^r as infinitesimals.[7] Ignore the ϵ_i terms for the time being. The first part on the right hand side of φ_i is derived from the characteristic function in equation (2) by using Theorem 1 and equation (3). To get the remaining term on the right hand side of φ_i consider the part of the characteristic function in equation (2) that applies to production. Since the expected value function is linear the expected contribution of a coalition is given by the sum of the expected contributions of its members. Rewriting the second part of the characteristic function in accordance with

[7] I owe this to Donald Katzner and Randall Bausor who pressed the point at a presentation to the University of Massachusetts Economics Department.

equation (4) the probability that any individual occurs in some interval of length t is given by t. From equation (3) this means that the expected value is

$$(1-a)\sum_{t=0}^{1} t(f \circ (\psi(t+dt)w(I) - f \circ (\psi t w(I))$$

where $dt = \mu$. It is now possible to use the mean value theorem and the inner-product identities to write the last term in brackets as

$$\langle \nabla f(z), \psi t w(I) \rangle \, dt \tag{5}$$

for some $z \in (\psi(t+dt)w(I), t\psi t w(I))$.

Since $\mu(S) = \frac{w^r(S)}{w^r(I)}$ with a probability as close to one as we wish and z is infinitesimally close to $t\psi w(I)$ equation (5) can be written

$$\langle \nabla f(z), \psi w(S) \rangle \, dt = f_{w(S)}(t\psi w(I))dt + o$$

where o is an error term which can be ignored. It is now possible to rewrite the expected value

$$(1-a)\int_0^1 \epsilon(S) f_{w(S)}(t\psi w(I))dt$$

and the proof follows by including the expression for the disutility of work and rearranging the directional derivative to give the value for agent i.

□

It will be noted that the theorem does not include the parameter, b, for the size of the coalition required in order to be to be decisive. This justifies the claim that the value of b does not change the value to any player.

3.3. Special case. No cost to working or everyone works

In this case an increase in the portion of the product allocated by political decision benefits an agent with less than average productivity and harms and agent with above average productivity. On the other hand it does not change the value of any coalition.

Corollary 1 of Theorem 2: Suppose that $\epsilon = \bar{\epsilon} = 1$ for all agents.
[a]. An increase in a benefits i if $\mu < w^r(i)$ and harms i otherwise.
[b] $\frac{\partial v(S)}{\partial a} - 0$.

Proof. [a]. Set $\epsilon_i = \bar{\epsilon}_i = 1$ and hence $\psi = 1$ use the fact that $f_{w^r} = f' w^r$. Now rewrite Theorem 2 as:

$$\varphi_i = f \circ (w(I))(a\mu + (1-a)w^r)$$

and differentiating with respect to a gives the result.
[b]. For a coalition the appropriate terms in the previous equation are $\mu(S)$ and $\sum w^r(S)$. Since $\mu(S) =$

$\sum w^r(S)$ the a term in the right hand brackets disappears as required. □

4 The effects of political power in the continuous approximation

4.1. The smooth approximation to the general problem

The case when individuals can choose not to work raises a number of questions about the effect of transfers by political decision on productivity and the structure of the characteristic function. From a slightly more applied perspective, these are also questions about how these transfers affect societies that start from different initial levels of equality in the distribution of productive capacities. Is the effect the same everywhere?

This case is more difficult to study. In order to simplify the problem think of the elements of w as numbers and rank them so that w can be treated as a step function that is monotonically increasing in r. Now construct a smooth function $\hat{w}$ with $\hat{w}_r > 0$ that interpolates w. To indicate this change in the analysis this is called the approximate system or the smooth system. To save notation the interpolating function is rewritten w.

In this set up ψ is also treated as a function that is continuously differentiable in a when we need to examine changes in the distribution mechanism.

4.2. Results for the smooth system

The consequences of Theorem 2 for the smooth system are roughly as follows. The portion of the agents that work either does not change or decreases as political power increases. What is more interesting is that the rate at which the set of non-productive agents changes for an increase in transfers through the political mechanism. This will depend on the variation in productive capacities amongst agents. Consider two systems with the same capacity to produce if all agents work. In this case the set of non-working agents will begin to increase at lower levels of transfer in the system with the less egalitarian initial distribution than in the system where this is more egalitarian. Beyond some point, however, the set of non-working agents will increase much more rapidly in the system with the more egalitarian distribution of initial endowments. In different words, the system with the more egalitarian initial distribution can sustain much higher levels of transfers without affecting productivity than the less egalitarian system. It is also shown that, if the spread in initial allocation of productive capacities is sufficiently large, total welfare for an additive function may initially increase as political power increases and then decrease.

4.3. Production and political power

In the first part of this corollary it is shown that total output either remains unchanged or decreases as political power increases, depending on the spread of endowmnents.

Corollary 2 of Theorem 2. Suppose $\epsilon = \bar{\epsilon} = x < 1$ if an agent works. Then:
[a]. $\psi_a < 0$ on some interval in a given by $(a_1, a_2) \in (0, 1)$ and $\psi_a = 0$ otherwise;
[b] If w^r sufficiently small there is some set of individuals A of measure $\mu(A) = c > 0$ who do not work for

any $a > 0$.

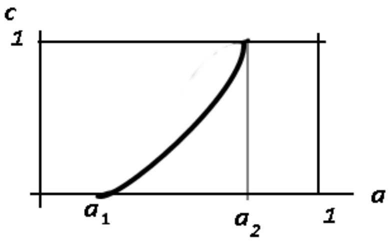

Figure 2. Example of a support for the set of non-working individuals.

Proof: Let $\hat{\varphi}_i$ be the value for an agent that does not work and define the value function

$$\zeta_i := \hat{\varphi}_i - \varphi_i$$

Then

$$\zeta_i = af \circ (\psi w(I))\mu(1 - \epsilon_i) - (1 - a)(\epsilon_i f \circ (\psi w(I))w^r \tag{6}$$

$[a]$. Assume to the contrary that $\psi_a > 0$ on some interval $(a_1, a_2) \in (0, 1)$ and the number of agents that work increases. In particular there must be some agent i with $\zeta_i = 0$ and $\zeta_{ia} < 0$. Differentiating equation (6) and evaluating at $\zeta_i = 0$ gives $\zeta_{ia} > 0$. This establishes the contradiction.

$[b]$ follows immediately from the definition of ζ_i
□

This still leaves the assertions about the relation between the spread of productive capacity and the rate of increase in the set of non productive individuals. These are covered in the following.

Corollary 2 of Theorem 2 continued: Consider ζ as a function of (w^r, a).
$[c]$. For $a = a' : 0 < c < 1$ and $\bar{r} = min\ r : \zeta(a, w^r) = 0$ with $\zeta > 0$ for all agents with capacities $w^r < w^{\bar{r}}$ we have $\bar{r}$ increasing in a;
$[d]$ As $|max\ w^r - min\ w^r|$ decreases the interval $[a_2 - a_1]$ that supports $0 \leq c \leq 1$ decreases.

Proof: $[c]$. First we need to show that there is a unique $\bar{r}$ for each $a : \zeta = 0$. Set $a = a' < 1$ where $a' : 0 < c < 1$. From equation (6) for $w^r = 0$ we have $\zeta > 0$ and for w^r sufficiently large we have $\zeta < 0$. From the intermediate value theorem there is at least one $r : \zeta = 0$. Since ζ is monotonically decreasing in r this $r = \bar{r}$ is unique.

Now let $y = (a', w^r) : \zeta = 0$ and consider the small ball β around y. Let $w^{r'} = w^r + x \in \beta$ for a' fixed. It follows from equation (6) that $\zeta(a', w^{r'}) < 0$. Taking the limit as $x \to 0$ we get $\partial\zeta/\partial w^r < 0$. Similarly, for w^r fixed we have $\partial\zeta/\partial a > 0$ in the vicinity of y. Using the implicit function theorem $\frac{dw^r}{da} = -\frac{\partial\zeta/\partial a}{\partial\zeta/\partial w^r}$ for all

$a, w^r \in \beta$ and hence $\frac{dw^r}{da} > 0$ in the interval $w^r, w^r + x$. By induction $\frac{dw^r}{da} > 0$ in the vicinity of the curve $(a, w^r) : \zeta = 0$. This is all we need to show $\bar{r}$ is increasing in a. See fig. 2.

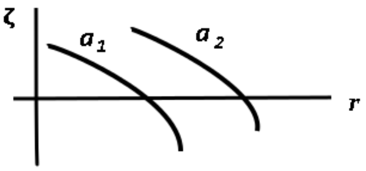

Figure 3. ζ as a function of (w^r, a) for $a_2 > a_1$

[d]. For $\zeta_i = 0$ we have $a = \frac{\epsilon w^r}{(\mu(1-\epsilon)+\epsilon w^r)}$. Let $maxw^r = \mu + x$ and $minw^r = \mu - x$ for some $x > 0$. Making the substitution and differentiating gives $\frac{\partial a}{\partial x} > 0$ for $x > 0$ and $\frac{\partial a}{\partial x} < 0$ for $x < 0$. Since (a_1, a_2) is an interval in a we are done.

□

It follows that if all individuals have the same productive capacity given by $w^r = \mu$ there is some $a' : c = 0$ for $a < a'$ and $c = 1$ for $a > a'$.

4.4 No loss of output conditions

It is often argued that distribution by governments is undesirable because of it reduces total output and we have seen some of this in the previous results. These show that distribution by voting does not necessarily reduce output because the interval $\bar{a} = (a_1, a_2)$ varies according to initial endowments. A question that remains is, what conditions would have to hold for production not to decrease over some interval in $(a'_1, a'_2) \subset \bar{a}$? In order to answer this question consider the parameter for the utility of work given by ϵ. This has been treated as fixed in relation to a. There is, however, also an argument that work does not always have negative utility and that in some cases work is valued for its own sake because it gives individuals an identity and carries status and so on. In this case the question is, could ϵ change so that an increase in distribution by voting does not always reduce output for $a \in \bar{a}$?

Considering the case where the utility from working may increase as initial endowments increases because those with more capacities or resources tend to get better jobs. It has been shown that as a increases the average value of w^r for the workforce in the complement of A increases. This means that we can capture this idea by treating the partial differential $\frac{\partial \epsilon_i}{\partial a} = \epsilon_{ia}$ as a proxy for the relation between job utility and distribution. We now have:

Corollary 3 of Theorem 2: There is a value of $e_a > 0$ such that c remains stationary as a increases in some interval $a \in \bar{a}$.

Proof. The corollary is the same as saying that $\zeta(a, \epsilon) = 0$ in some interval in $a \in \bar{a}$. This means that there is a manifold $\beta(a, \epsilon) = a\mu - \epsilon_i(a\mu + (1 - a)w^r) = 0$. From the implicit function theorem we have

$$\epsilon_a = \frac{\mu - \epsilon(\mu + w^r_a(1-a) - w^r)}{a\mu + w^r(1-a)}$$

and this is the condition that satisfies the corollary. □

We also know that under the assumption that the initial allocation changes fairly evenly between its highest and lowest values for each type of system and $\sum w^r(I) = 1$ that w^r must increase more rapidly for the more egalitarian than for the less egalitarian system. Taking the derivative of ϵ_a gives

$$\frac{\partial \epsilon_a}{\partial w^r_a} < 0$$

It follows from this that the condition required for c to remain stationary in some interval $a \in \bar{a}$ can be satisfied for lower levels of change in the utility for work in the system with the more egalitarian initial distribution. In other word, the condition that output does not fall in some interval in $\bar{a}$ is more likely to be met in the more egalitarian system.

4.5 Welfare

It was claimed that, for an additive welfare function, and a sufficiently large spread of initial capacities aggregate welfare increases and then decreases as a increases. Write the welfare function

$$\vartheta = \int_0^c \hat{\varphi} + \int_c^1 \varphi \tag{7}$$

Corollary 4 of Theorem 2: Let $w^{\bar{r}} = min\ w^r$. For $w^{\bar{r}}$ sufficiently small there is an ϵ sufficiently small that ϑ increases for $a \in (0, s)$ for some $s < 1$ and decreases for $a \in (s', 1)$ for $s' > s$;

Proof: $[a]$. Differentiating equation (7) gives

$$\vartheta_a = \int_0^c \hat{\varphi}_a\ dt + \hat{\varphi}(c)\frac{\partial c}{\partial a} + \int_c^1 \varphi_a\ dt - \varphi(c)\frac{\partial c}{\partial a}$$

and expanding gives

$$\vartheta_a = \int_0^c f\mu dt + \epsilon \int_c^1 f\mu dt - \epsilon \int_c^1 fw^r + af(c)\frac{\partial c}{\partial a}\mu(c) - a\epsilon f(c)\frac{\partial c}{\partial a}\mu(c) - (1-a)\epsilon f(c)\frac{\partial c}{\partial a}w^r(c) + Df\ terms$$

where

$$Df = a\int_0^c \frac{\partial f}{\partial c}\mu dt + \epsilon a \int_c^1 \frac{\partial f}{\partial c}\mu dt + (1-a)\epsilon \int_c^1 \frac{\partial f}{\partial c} w^r$$

Since $\mu(S) - w^r(S) = 0$ we have $\epsilon\mu \int_c^1 f dt - \epsilon \int_c^1 f w^r = 0$ and the second and third terms cancel out. Now set ϵ sufficiently small that all terms except $\int_0^c f\mu dt$, $af(c)\frac{\partial c}{\partial a}\mu(c)$ and $a\int_0^c \frac{\partial f}{\partial c}\mu(c)dt$ become negligible. To take care of the negative term set min w^r sufficiently small that $\left|\int_0^c (f + a\frac{\partial f}{\partial c})\mu dt\right| < \iota$ for some ι sufficiently small that $\vartheta_a > 0$. To show there is an $s > 0$ such that $\vartheta_a < 0$ observe that as c increases $\frac{\partial f}{\partial c}$ decreases.
□

Example. Consider a discrete case. Suppose we only have two types $r = \{1, 2\}$ and construct smooth system as before. Then

$$\vartheta = af\mu + \epsilon af\mu + \epsilon(1-a)fw^2$$

and differentiating with respect to a around $a : \zeta_1 = 0$ gives $\vartheta_a = f\mu(1+\epsilon) - \epsilon fw^2 + Df$. If w^1 is sufficiently small f_a is small and for $\epsilon w^2 < \mu$ and ϵ sufficiently small $\vartheta_a > 0$.

An implication of this is that an increase in the distribution by voting may increase total welfare in a system that is highly inegalitarian and working conditions are bad. In a more egalitarian system this increase will make those with less than average endowments better off, but will not change total welfare. In this case the story is the same as in the first corollary.

5 Conclusion

This paper constructed a political-economic model that might help understand some questions about production and distribution in an alternative framework to micro-foundations approaches. It showed among other things that, when individuals cannot withdraw their labour, the value to an individual with low levels of productivity increases as political power increases but the total value of any coalition is not altered. When individuals can withdraw their labour a more egalitarian system can sustain higher levels of transfers without any loss in total welfare. On the other hand a more egalitarian system exhibits a sort of threshold over which the total welfare costs of distribution begin to escalate rapidly.

From these and other results in the paper it could be argued that it may not be prudent to simply assume there is an inverse relation between output and the portion of the product transfered by political decision. Not only might the situation be more complicated, but over some range of transfers and initial conditions the affect on output may be negligible.

It was also shown that government transfers can be welfare improving if the extremes of initial endowments in productive capacities are large. If productive capacities are roughly equal government transfers will be neutral up to some point and reduce welfare after that point.

In addition the continuous agent approach raises some modelling questions. In the first place it is not clear that a tendency to over-rationalize in trying to analyze political-economic systems in a formal framework has not developed in some work. It is not at all clear that voters and political entrepreneurs have the sort of calculating capacity that many models require.

I also think, secondly, that advances in communications technologies might allow increasing fragmentation and targeting of voters. If so the assumption that any possible coalitions can form may become more important. For both reasons models built around individual optimizing behavior may miss important features of distribution questions.

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